Journal of Symbolic Computation 44 (2009) 1374–1393

Contents lists available at ScienceDirect

Journal of Symbolic Computation

journal homepage: www.elsevier.com/locate/jsc

Letterplace ideals and non-commutative Gröbner bases

Roberto La Scala[a,1], Viktor Levandovskyy[b]

[a] *Dipartimento di Matematica, via Orabona 4, 70125 Bari, Italie*

[b] *RWTH Aachen, Templergraben 64, 52062 Aachen, Germany*

ARTICLE INFO

Article history:
Received 28 May 2008
Accepted 18 March 2009
Available online 5 April 2009

To our beloved daughters, Bohdana, Irene, Agata

Keywords:
Free associative algebras
Gröbner bases
Monoid action
Invariant ideals

ABSTRACT

In this paper we propose a 1-to-1 correspondence between graded two-sided ideals of the free associative algebra and some class of ideals of the algebra of polynomials, whose variables are double-indexed commuting ones. We call these ideals the "letterplace analogues" of graded two-sided ideals. We study the behaviour of the generating sets of the ideals under this correspondence, and in particular that of the Gröbner bases. In this way, we obtain a new method for computing non-commutative homogeneous Gröbner bases via polynomials in commuting variables. Since the letterplace ideals are stable under the action of a monoid of endomorphisms of the polynomial algebra, the proposed algorithm results in an example of a Buchberger procedure "reduced by symmetry". Owing to the portability of our algorithm to any computer algebra system able to compute commutative Gröbner bases, we present an experimental implementation of our method in Singular. By means of a representative set of examples, we show finally that our implementation is competitive with computer algebra systems that provide non-commutative Gröbner bases from classical algorithms.

1. Introduction

Since the very beginning of abstract algebra, scientists computed examples in complicated algebraic structures like semigroups, groups, rings and algebras. It became clear that one has to use universal objects like free algebras and present the specific objects of some class as free objects subject to some relations. Since that time algebraists used – at first implicitly – rewriting rules based on

E-mail addresses: lascala@dm.uniba.it (R. La Scala), levandov@math.rwth-aachen.de (V. Levandovskyy).

1 Tel.: +39 080 544 2674; fax: +39 080 544 3610.

0747-7171/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jsc.2009.03.002

the relations and completion of the relations. Gradually a need to write these algorithms down in a computer science fashion, with proofs of termination (or partial termination) and correctness was widely recognized.

A pioneering paper of Shirshov (1962) dealt with free Lie algebras. The thesis of Buchberger in 1965 (Buchberger, 2006) laid down the foundation of commutative computer algebra, and heavily influenced the non-commutative theory as well. The famous Diamond lemma of Bergman (1978) was formulated in such a way that one could apply it to various algebraic structures including free associative algebras.

It has taken some time for non-commutative Gröbner basis theory to develop in the ways proposed by Buchberger and Bergman. Then, Mora in (1986; 1989; 1988), Green in (1993; 2000) and Ufnarovski (1995, 1998) and Cojocaru et al. (1997) presented different facets of what we call today non-commutative Gröbner basis theory. In particular Mora (1988) discussed free non-commutative algebras and their quotient rings endowed also with negative (non-well-)orderings and (Apel, 2000) further extended his theory. Other important contributions were made by Apel and Lassner (1988) and especially Apel (2000).

Pritchard (1996) investigated weak Gröbner bases and the ideal membership problem in free associative algebras with commutative rings as coefficient domains. Leonid Bokut' with his colleagues developed Gröbner–Shirshov basis theory (see e.g. Bokut and Klein (1996)) and applied it to numerous famous algebraic structures. In a recent paper (Kang et al., 2007), the authors provided a generalization of the celebrated F_4 algorithm of J.-C. Faugère in the framework of Gröbner–Shirshov basis theory. However, their promising algorithm is not known to be implemented in any computer algebra system.

The computer algebra systems Grb (later superseded by OPAL (Green et al., 1997; Keller, 1997)) and Bergman (Backelin et al., 2006) were developed in the groups of Ed Green, and Backelin and Ufnarovski respectively. Also, several important classes of non-commutative algebras were distinguished and special Gröbner basis theories were developed for them; see e.g. Kandri-Rody and Weispfenning (1990), Apel (1988), Li (2002), Kredel (1993), Pesch (1998), Weispfenning (1992), Bueso et al. (2003), Levandovskyy (2005a), Levandovskyy (2005b) and Chyzak and Salvy (1998). The latter cases are often accompanied with implementations which do not really compute in free associative algebras, but utilize special properties of algebras in most aspects, from particular presentation of data to simplified forms of generalizations of Buchberger's algorithm.

The Gröbner basis theory for monoid and group rings was developed by Reinert (1995) and Rosenmann (1993). Both Madlener with Reinert and Heyworth (Heyworth, 2000) pointed out the fact that rewriting for semigroups (groups) is a special case of Gröbner basis theory.

In the last few years we have seen more progress in both theoretical, implementational and practical directions. Notably, the interest in free associative algebras grew stronger, as indicated by e.g. the book of Green (2003), where the author considers also negative (non-well-)orderings for certain non-commutative cases with a very different motivation and meaning, compared with the theory of Mora (1988) and Apel (2000) and with the commutative case as in Greuel et al. (2002).

Evans and Wensley investigated in Evans and Wensley (2007) involutive bases in non-commutative algebras.

Gröbner bases are applied to a wide area of problems, starting with representation theory of groups and (Lie) algebras, to algebraic D-modules, control theory, symbolic summation, integration and many more. In applications one needs the flexibility of ground fields or rings, possible monomial (module) orderings and different kinds of Gröbner bases. The most fundamental applications of Gröbner bases, called *Gröbner basics* by Buchberger and Sturmfels, give rise to non-commutative Gröbner basics. The tools of homological algebra, such as syzygies and projective resolutions, become more and more essential even for the problems of applied nature, since they lead to e.g. extension modules and various kinds of cohomologies.

Keeping all this in mind, in this paper we propose a new method for computing Gröbner bases of graded two-sided ideals by means of an embedding of all computations in a commutative polynomial ring. Let $K\langle X\rangle$ be the associative algebra freely generated by a countable or finite set X, and denote by $K[X|P]$ the commutative polynomial ring whose variables belong to the set $X \times P$, where $P = \mathbb{N}$ is the set of natural numbers. We consider over the algebra $K[X|P]$ a direct sum decomposition into subspaces and an action by endomorphisms both defined by the monoid $\mathbb{N}$. To be precise, we define the *shift* $s \in \mathbb{N}$ of a monomial $m = (x_{i_1}|j_1)\cdots(x_{i_n}|j_n)$ as the minimum of the indices j and we put $s \cdot m = (x_{i_1}|s+j_1)\cdots(x_{i_n}|s+j_n)$.

In Section 2 we start with an embedding of vector spaces $\iota : K\langle X\rangle \to K[X|P]$ where $\iota(w) = (x_{i_1}|0)\cdots(x_{i_n}|n-1)$ for any monomial $w = x_{i_1}\cdots x_{i_n}$ of $K\langle X\rangle$. This map was first introduced in Feynman (1951) and Doubilet et al. (1974) for the purposes of representation theory. By means of it, we obtain a 1-to-1 correspondence $\tilde{\iota}$ between all graded (w.r.t. total degree) two-sided ideals $I \subset K\langle X\rangle$ and a special class of ideals $J \subset K[X|P]$ that we call *letterplace ideals*. Such an ideal satisfies some set of natural properties and in particular J is invariant with respect to the shift action. In Section 3 we study how homogeneous bases of I and J behave under the correspondence, and in particular Gröbner bases. We prove that a homogeneous Gröbner basis G of the two-sided ideal I can be obtained as $\iota^{-1}(H \cap V)$, where H is a suitable Gröbner basis of J and V is the image of ι.

For the purpose of this result, it is clear that we have to consider compatibility conditions with respect to the map ι for the term-orderings of the algebras $K\langle X\rangle$, $K[X|P]$. Since the letterplace ideals are shift-invariant, a first assumption for a term-ordering $\prec$ of $K[X|P]$ is to be such that $u \prec v$ if and only if $s \cdot u \prec s \cdot v$ for any monomials u, v and shift s. This implies that for an S-polynomial one has $S(s\cdot f, t\cdot g) = s \cdot S(f, (t-s)\cdot g)$ for all shifts $s \le t$ and f, g polynomials, which means that we have reduced to considering S-polynomials with lowest shift (zero). In other words, Buchberger's criterion and his completion procedure can be reduced up to the symmetry defined by the shift action. Clearly this approach works not only for letterplace ideals but also for all shift-invariant ones.

Finally, to obtain the Gröbner basis of I, we are interested just in the elements belonging to the intersection $H \cap V$. Hence, the last ingredient is finding for the (reduced by symmetry) Buchberger procedure a criterion that deletes all S-polynomials leading to unnecessary elements of the Gröbner basis H of J. The resulting algorithm shows its feasibility in Section 5, where we compare an experimental implementation in the computer algebra system Singular with four of the best implementations of non-commutative Gröbner bases, namely the ones of Bergman, GAP, Magma and Opal. In our test set we consider relevant objects in non-commutative algebra, as two-sided ideals defining the universal enveloping algebras of some relatively free Lie algebras, Serre's relations (built from generalized Cartan matrix) between positive or negative roots of Lie algebras and of Kac–Moody algebras, and some braid-like algebras. We also consider the T-ideal of the polynomial identities satisfied by a matrix algebra. Owing to the naturalness of the proposed approach, we aim in the future to enlarge the range of its applications to all non-commutative Gröbner basics, and also to extend the idea of an optimization of the Buchberger procedure for invariant ideals to actions of other types. We hope finally that the portability of our algorithm in any computer algebra system providing commutative Gröbner bases will extend the number of systems able to support the work of researchers in non-commutative algebra.

2. A bijection between ideals

Let K be any field. Fix $X = \{x_0, x_1, \ldots\}$ a finite or countable set and put $P = \mathbb{N} = \{0, 1, \ldots\}$. We call X and P respectively *sets of letters and places*. We denote as $(x_i|j)$ each element (x_i, j) of the product set $X \times P$. Define $K\langle X\rangle$, the free associative algebra generated by X, and denote by $K[X|P]$ the polynomial ring in the commuting variables $(x_i|j)$. Let $\langle X\rangle$ and $[X|P]$ be the monoids given by the corresponding sets of monomials. Let $\mu = (\mu_k)_{k\in\mathbb{N}}$, $\nu = (\nu_k)_{k\in\mathbb{N}}$ be two sequences of non-negative integers with finite support. We can consider μ as a multidegree for the monomials $w = x_{i_1}\cdots x_{i_n} \in \langle X\rangle$ and (μ, ν) as a multidegree for the monomials

$$m = (x_{i_1}|j_1)\cdots(x_{i_n}|j_n) \in [X|P].$$

To be precise, we define $\mu_k = \#\{\alpha \mid x_{i_\alpha} = x_k\}$, $\nu_k = \#\{\beta \mid j_\beta = k\}$ and define

$$\partial_X(m) = \mu, \partial_P(m) = \nu.$$

Defining $|\mu| = \sum_k \mu_k$, one has that $|\partial_X(m)| = |\partial_P(m)| = \deg(m)$. We call μ and ν respectively *letter- and place-multidegrees*. We denote by $K\langle X\rangle_\mu$ the homogeneous component of the algebra $K\langle X\rangle$ corresponding to the letter-multidegree μ, that is $K\langle X\rangle_\mu$ is the subspace of $K\langle X\rangle$ spanned by all monomials of multidegree μ. In the same way we can define the homogeneous component $K[X|P]_{\mu,\nu}$

and hence $K\langle X\rangle = \bigoplus_{\mu} K\langle X\rangle_{\mu}$ and $K[X|P] = \bigoplus_{\mu,\nu} K[X|P]_{\mu,\nu}$ are multigraded algebras. By putting

$$K[X|P]_{*,\nu} = \bigoplus_{\mu} K[X|P]_{\mu,\nu}, \; K[X|P]_{\mu,*} = \bigoplus_{\nu} K[X|P]_{\mu,\nu}$$

we obtain that $K[X|P]$ is multigraded with respect to letter- or place-multidegrees only. Clearly $K\langle X\rangle$, $K[X|P]$ are also graded algebras with respect to total degrees (we always assume all variables have degree 1).

The monoid $\mathbb{N}$ has a natural faithful action on the graded algebra $K[X|P]$ given by

$$s \cdot (x_i|j) = (x_i|s+j)$$

for all variables $(x_i|j)$ of $K[X|P]$ and $s \in \mathbb{N}$. In other words, one has a monoid monomorphism $\rho : \mathbb{N} \to \mathrm{End}(K[X|P])$ where each map $\rho(s)$ is homogeneous of degree zero. Note also that the $\rho(s)$ are injective maps.

Definition 2.1. Let $m = (x_{i_1}|j_1)\cdots(x_{i_n}|j_n)$ be a monomial of $[X|P]$. We define $\mathrm{sh}(m) = \min\{j_1,\ldots,j_n\}$ and we call this integer the *shift of m*.

If $\partial_P(m) = \nu$ we have clearly that $\mathrm{sh}(m) = \min\{k \mid \nu_k > 0\}$. Moreover one has

$$\mathrm{sh}(m_1m_2) = \min\{\mathrm{sh}(m_1), \mathrm{sh}(m_2)\}$$

for all $m_1, m_2 \in [X|P]$. Denote by $K[X|P]^{(s)}$ the subspace of $K[X|P]$ generated by all monomials with shift $s \in \mathbb{N}$. We call the elements of $K[X|P]^{(s)}$ *shift-uniform with shift s*. One has that $K[X|P] = \bigoplus_{s\in\mathbb{N}} K[X|P]^{(s)}$ and $s \cdot K[X|P]^{(t)} = K[X|P]^{(s+t)}$ for all $s, t \in \mathbb{N}$.

For each $s, n \in \mathbb{N}$ we denote by $s \cdot 1^n$ the place-multidegree $\nu = (\nu_k)_{k\in\mathbb{N}}$ such that

$$\nu_k = \begin{cases} 1 & \text{if } s \le k \le s+n-1, \\ 0 & \text{otherwise.} \end{cases}$$

For $s = 0$ we write simply 1^n. Define now

$$V = \bigoplus_{n\in\mathbb{N}} K[X|P]_{*,1^n}$$

which is a subspace of $K[X|P]^{(0)}$. We have a useful isomorphism of vector spaces

$$\iota : K\langle X\rangle \to V \quad w \mapsto (x_{i_1}|0)\cdots(x_{i_n}|n-1)$$

for any monomial $w = x_{i_1}\cdots x_{i_n} \in \langle X\rangle$. Clearly ι preserves letter-multidegrees and hence total degrees. Since $K\langle X\rangle$ is isomorphic to the tensor algebra over the vector space KX with basis X, a monomial w can be understood as a decomposable tensor. For this reason the image $(x_{i_1}|0)\cdots(x_{i_n}|n-1)$ is called the *letterplace notation of the tensor w*. The isomorphism ι has been introduced in the context of invariant and representation theory (see Doubilet et al. (1974), Section 10, and Feynman (1951)). Note that the general linear group $\mathrm{GL}_m(K)$, where $m = \#X$, and the symmetric group S_n act respectively from the left and the right on both spaces $K\langle X\rangle_n = \bigoplus_{|\mu|=n} K\langle X\rangle_{\mu}$ and $V_n = \bigoplus_{|\mu|=n} K[X|P]_{\mu,1^n}$. Then, the restricted isomorphism $\iota_n : K\langle X\rangle_n \to V_n$ results in an isomorphism of modules.

Definition 2.2. Let J be an ideal of $K[X|P]$. The ideal J is called:

- *place-multigraded* if $J = \sum_{\nu} J_{*,\nu}$ where $J_{*,\nu} = J \cap K[X|P]_{*,\nu}$,
- *shift-decomposable* if $J = \sum_s J^{(s)}$ where $J^{(s)} = J \cap K[X|P]^{(s)}$.

Proposition 2.3. *Let $J \subset K[X|P]$ be an ideal. Then J is shift-decomposable if and only if J is generated by $\bigcup_{s\in\mathbb{N}} J^{(s)}$.*

Proof. The necessary condition is obvious. Assume now that $J = \mathrm{Span}(mf \mid m \in [X|P], f \in J^{(s)}, s \in \mathbb{N})$. Then, for $t = \min\{\mathrm{sh}(m), s\}$ we have $mf \in J^{(t)}$ and hence $J = \sum_s J^{(s)}$. □

Clearly a place-multigraded ideal is also graded and shift-decomposable.

Definition 2.4. Let J be a shift-decomposable ideal of $K[X|P]$. We say that J is *shift-invariant* if $s \cdot J^{(t)} = J^{(s+t)}$ for all $s, t \in \mathbb{N}$.

Clearly J is shift-invariant if and only if $s \cdot J^{(0)} = J^{(s)}$, since in this case $s \cdot J^{(t)} = s \cdot (t \cdot J^{(0)}) = (s+t) \cdot J^{(0)} = J^{(s+t)}$.

Proposition 2.5. *Let $J \subset K[X|P]$ be an ideal. Then J is shift-invariant if and only if $J = \sum_{s \in \mathbb{N}} s \cdot J^{(0)}$.*

Proof. Clearly we have the necessary condition. Assume now $J = \sum_s s \cdot J^{(0)}$. We have $s \cdot J^{(0)} \subset J$ and $s \cdot J^{(0)} \subset s \cdot K[X|P]^{(0)} = K[X|P]^{(s)}$ and hence $s \cdot J^{(0)} \subset J^{(s)}$. Let $f \in J^{(s)}$. Since f is shift-uniform and $J = \sum_t t \cdot J^{(0)}$, then necessarily $f \in s \cdot J^{(0)}$. We conclude that $s \cdot J^{(0)} = J^{(s)}$ and therefore $J = \sum_s J^{(s)}$. □

Proposition 2.6. *Let J be an ideal of $K[X|P]$ and put $I = \iota^{-1}(J \cap V) \subset K\langle X\rangle$.*

- *If J is a shift-invariant ideal, then I is a left ideal of $K\langle X\rangle$.*
- *If J is a place-multigraded ideal, then I is a graded right ideal.*

Proof. Assume J is shift-invariant and let $f \in I$, $w \in \langle X\rangle$. Define $g = \iota(f) \in J \cap V$ and $m = \iota(w)$. If $\deg(w) = s$ we have clearly $\iota(wf) = m(s \cdot g) \in J \cap V$ and therefore $wf \in I$. Suppose now that J is place-multigraded and hence graded. Since V is a graded subspace, it follows that $J \cap V = \sum_d (J_d \cap V)$ and then, setting $I_d = \iota^{-1}(J_d \cap V)$, we obtain $I = \sum_d I_d$. Let $f \in I_d$, that is $\iota(f) = g \in J_d \cap V$. For all $w \in \langle X\rangle$ we have that $\iota(fw) = g(d \cdot m) \in J \cap V$, that is $fw \in I$. □

Proposition 2.7. *Let I be a left ideal of $K\langle X\rangle$ and put $I' = \iota(I)$. Define J, the ideal of $K[X|P]$ generated by $\bigcup_{s \in \mathbb{N}} s \cdot I'$. Then J is a shift-invariant ideal. Moreover, if I is graded then J is place-multigraded.*

Proof. From $s \cdot I' \subset J^{(s)}$ it follows that J is generated by $\bigcup_{s \in \mathbb{N}} J^{(s)}$, that is J is shift-decomposable. By definition one has $J = \mathrm{Span}(m(t \cdot f) \mid m \in [X|P], t \in \mathbb{N}, f \in I')$. Then, the vector space $J^{(s)}$ is spanned by elements $m(t \cdot f)$ such that $\min\{\mathrm{sh}(m), t\} = s$. In particular, $J^{(0)}$ is spanned by elements $m(t \cdot f)$ where $\min\{\mathrm{sh}(m), t\} = 0$. By acting with s, we obtain that $s \cdot J^{(0)}$ is spanned by elements of the form $s \cdot (m(t \cdot f)) = (s \cdot m)((s+t) \cdot f)$ where $m \in [X|P], t \in \mathbb{N}, f \in I'$ such that $\min\{\mathrm{sh}(m), t\} = 0$ and therefore $\min\{\mathrm{sh}(s \cdot m), s+t\} = s$. Since $s \cdot K[X|P]^{(0)} = K[X|P]^{(s)}$ we conclude that $s \cdot J^{(0)} = J^{(s)}$.

Assume now that I is a graded ideal. Any element $f \in I$ can be written as $f = \sum_d f_d$ where $f_d \in I \cap K\langle X\rangle_d$. Put $g = \iota(f)$, $g_d = \iota(f_d)$ and then $g_d \in I' \cap V_d$. For any $s \in \mathbb{N}$ one has that $s \cdot g = \sum_d s \cdot g_d$ where $s \cdot g_d \in s \cdot (I' \cap V_d) \subset J$. Note that since $\partial_P(s \cdot g_d) = s \cdot 1^d$, all polynomials $s \cdot g_d$ are homogeneous with respect to place-multigrading. We conclude that J is generated by homogeneous elements and hence it is a place-multigraded ideal. □

Definition 2.8.

- Let $I \subset K\langle X\rangle$ be a graded two-sided ideal. We denote by $\tilde{\iota}(I)$ the shift-invariant place-multigraded ideal $J \subset K[X|P]$ generated by $\bigcup_{s \in \mathbb{N}} s \cdot \iota(I)$ (cf. Proposition 2.7), and call J *the letterplace analogue of the ideal I*.
- For a shift-invariant place-multigraded ideal $J \subset K[X|P]$ we denote by $\tilde{\iota}^{-1}(J)$ the graded two-sided ideal $I = \iota^{-1}(J \cap V) \subset K\langle X\rangle$ (cf. Proposition 2.6).

Proposition 2.9. *We have the following inclusions:*

- $\tilde{\iota}^{-1}(\tilde{\iota}(I)) = I$, $\tilde{\iota}(\tilde{\iota}^{-1}(J)) \subseteq J$,
- $\tilde{\iota}(\tilde{\iota}^{-1}(J)) = J$ *if and only if J is generated by $\bigcup_{s,d \in \mathbb{N}} s \cdot (J_d \cap V)$.*

Proof. Let I be a graded two-sided ideal of $K\langle X\rangle$ and put $J = \tilde{\iota}(I)$, that is $J \subset K[X|P]$ is the ideal generated by $\bigcup_s s \cdot I'$ where $I' = \iota(I)$. We have to prove $I = \iota^{-1}(J \cap V)$, that is $I' = J \cap V$. Clearly $I' \subset J \cap V$. Let us have $f' \in J \cap V$ and put $f = \iota^{-1}(f')$. Since J is place-multigraded, we can assume that f' is homogeneous with respect to place-multigrading. Moreover we have $J = \mathrm{Span}(m(s \cdot h) \mid m \in [X|P], s \in \mathbb{N}, h \in I')$ and then it is sufficient to consider $f' = m(s \cdot h)$ for some m, s, h such that $f' \in V$. If $d = \deg(h)$ then $\partial_P(s \cdot h) = s \cdot 1^d$ and we can write $f' = m_1(s \cdot h)((s+d) \cdot m_2)$ with $m_1, m_2 \in [X|P] \cap V$ ($s = \deg(m_1)$). This implies that $f = w_1 g w_2$ where $g = \iota^{-1}(h)$, $w_i = \iota^{-1}(m_i)$. From $g \in I$ it follows that $f \in I$ (that is $f' \in I'$) and we conclude that $I = \tilde{\iota}^{-1}(J)$.

Now, let J be a shift-invariant place-multigraded ideal of $K[X|P]$. We put $I = \tilde{\iota}^{-1}(J)$, that is $I = \iota^{-1}(J \cap V)$. Thus $I' = \iota(I) \subset J$ and hence $\bigcup_s s \cdot I' \subset J$ because J is shift-invariant. We conclude that the ideal $\tilde{\iota}(I) \subset K[X|P]$ generated by $\bigcup_s s \cdot I'$ is contained in J. Since J is place-multigraded one has that $I' = J \cap V = \sum_d J_d \cap V$. We obtain that $\tilde{\iota}(I) = J$ if and only if J is generated by $\bigcup_{s,d} s \cdot (J_d \cap V)$. $\square$

Definition 2.10. A graded ideal J of $K[X|P]$ is called a *letterplace ideal* if J is generated by $\bigcup_{s,d\in\mathbb{N}} s \cdot (J_d \cap V)$. In this case J is shift-invariant and place-multigraded.

After this definition, we obtain finally the main result of this section.

Corollary 2.11. *The map $\iota : K\langle X\rangle \to V$ induces a one-to-one correspondence $\tilde{\iota}$ between graded two-sided ideals I of the free associative algebra $K\langle X\rangle$ and letterplace ideals J of the polynomial ring $K[X|P]$.*

3. Generating sets

This section concerns the problem of understanding how generating sets behave under the ideal correspondence $\tilde{\iota}$. We start with general bases, and we then consider Gröbner bases.

Definition 3.1. Let J be a letterplace ideal of $K[X|P]$ and $H \subset K[X|P]$. We say that H is a *letterplace basis* of J if $H \subset \bigcup_{d\in\mathbb{N}} J_d \cap V$ and $\bigcup_{s\in\mathbb{N}} s \cdot H$ is a generating set of the ideal J.

Proposition 3.2. *Let I be a graded two-sided ideal of $K\langle X\rangle$ and put $J = \tilde{\iota}(I)$. Moreover, let $G \subset \bigcup_{d\in\mathbb{N}} I_d$ and define $H = \iota(G) \subset \bigcup_{d\in\mathbb{N}} J_d \cap V$. Then G is a generating set of I as two-sided ideal if and only if H is a letterplace basis of J.*

Proof. Assume that $\bigcup_{s\in\mathbb{N}} s \cdot H$ is a basis of J, that is $J = \mathrm{Span}(m(s \cdot h) \mid m \in [X|P], s \in \mathbb{N}, h \in H)$. Since J is place-multigraded one has that

$$J \cap V = \mathrm{Span}(m(s \cdot h) \in V \mid m \in [X|P], s \in \mathbb{N}, h \in H).$$

If $d = \deg(h)$ then $m(s \cdot h) = m_1(s \cdot h)((s + d) \cdot m_2)$ where $m_1, m_2 \in [X|P] \cap V$ ($s = \deg(m_1)$). By applying ι^{-1} we obtain that $I = \mathrm{Span}(w_1 g w_2 \mid w_1, w_2 \in \langle X\rangle, g \in G)$, that is G is a basis of I as two-sided ideal. Assume now G generates I. By reversing the above argument, one has that $J \cap V \subset U = \mathrm{Span}(m(s \cdot h) \mid m \in [X|P], s \in \mathbb{N}, h \in H) \subset J$. From $s \cdot (m(t \cdot h)) = (s \cdot m)((s + t) \cdot h)$ for $s, t \in \mathbb{N}$, it follows that $s \cdot (J \cap V) \subset U$ for any s. We conclude that $J = U$ (that is H is a letterplace basis of J) because J is generated by $\bigcup_{s\in\mathbb{N}} s \cdot (J \cap V)$. $\square$

We want now to enter into the realm of Gröbner bases and we need for this purpose some new notions. Let X, Y be finite or countable sets (for the applications that we are interested in, one has $Y = X \times P$) and form the polynomial algebras $K\langle X\rangle$ and $K[Y]$. Denote by A either $K\langle X\rangle$ or $K[Y]$, and by M the monoid of all monomials of A. A (global) *term-ordering* on A is a total order on M which is a multiplicatively compatible well-ordering. To be precise, one has:

(i) either $u \prec v$ or $v \prec u$, for any $u, v \in M$, $u \neq v$;
(ii) if $u \prec v$ then $wu \prec wv$ and $uw \prec vw$, for all $u, v, w \in M$;
(iii) every non-empty subset of M has a minimal element.

Note that there are term-orderings even if the number of variables of the polynomial algebra A is infinite. In fact, by Higman's lemma (Higman, 1952), one has that any multiplicatively compatible total ordering on M such that $1 \prec x_0 \prec x_1 \prec \cdots$ is a term-ordering (see also Aschenbrenner and Pong (2004)).

If $f \neq 0$ is a polynomial in A we denote by $\mathrm{lm}(f)$ the greatest monomial of f with respect to a fixed term-ordering, and by $\mathrm{lc}(f) \in K \setminus \{0\}$ its coefficient. If G is any subset of $K\langle X\rangle$ we put $\mathrm{lm}(G) = \{\mathrm{lm}(g) \mid g \in G, g \neq 0\}$ and define $\mathrm{LM}(G)$ to be the two-sided ideal generated by $\mathrm{lm}(G)$. Let I be a two-sided ideal of $K\langle X\rangle$ and $G \subset I$. If $\mathrm{lm}(G)$ is a generating set of $\mathrm{LM}(I)$ then G is called a *Gröbner basis of I as two-sided ideal*. In other words, for all $f \in I, f \neq 0$ there are $w_1, w_2 \in \langle X\rangle, g \in G \setminus \{0\}$ such that $\mathrm{lm}(f) = w_1 \,\mathrm{lm}(g) w_2$. By induction on the term-ordering one easily obtains that Gröbner bases are in fact bases of ideals. Similarly one defines the notion of the Gröbner basis of an ideal of $K[Y]$.

Let $G \subset K[Y], f \in K[Y]$. By definition f has a *Gröbner representation with respect to G*, if $f = 0$ or there are $f_i \in K[Y], g_i \in G$ such that $f = \sum_{i=1}^{n} f_i g_i$, with either $f_i g_i = 0$ or $\mathrm{lm}(f) \succeq \mathrm{lm}(f_i)\,\mathrm{lm}(g_i)$ for all i. It is useful to introduce the following procedure.

Algorithm 1 REDUCE

Input: $G \subset K[Y]$ and $f \in K[Y]$.
Output: $h \in K[Y]$ such that $h = 0$ or $\mathrm{lm}(h) \notin \mathrm{LM}(G)$.
$h := f$;
while $h \neq 0$ and $\mathrm{lm}(h) \in \mathrm{LM}(G)$ **do**
  choose $g \in G, g \neq 0$ such that $\mathrm{lm}(g) \mid \mathrm{lm}(h)$;
  $h := h - \frac{\mathrm{lc}(h)\,\mathrm{lm}(h)}{\mathrm{lc}(g)\,\mathrm{lm}(g)} g$;
end while;
return h.

By $\mid$ we mean the usual divisibility of monomials. The termination of REDUCE is provided by the properties of term-orderings. Note in particular that even if G is infinite there are a finite number of monomials $\mathrm{lm}(g) \preceq \mathrm{lm}(f)$ with $g \in G, g \neq 0$. By induction on the term-ordering, one can easily prove the following characterizations.

Lemma 3.3.

- *The polynomial f has a Gröbner representation with respect to G if and only if* REDUCE$(f, G) = 0$.
- *The subset G is a Gröbner basis of the ideal $J \subset K[Y]$ if and only if any element $f \in J$ has a Gröbner representation with respect to G.*

For $f_1, f_2 \in K[Y] \setminus \{0\}, f_1 \neq f_2$ we put $m_i = \mathrm{lm}(f_i)$, $c_i = \mathrm{lc}(f_i)$ and $m = \mathrm{lcm}(m_1, m_2)$. We define

$$S(f_1, f_2) = \frac{m}{c_1 m_1} f_1 - \frac{m}{c_2 m_2} f_2.$$

The element $S(f_1, f_2)$ is called the *S-polynomial of the pair* (f_1, f_2). Clearly one has $S(f_2, f_1) = -S(f_1, f_2)$. Moreover, if m_1, m_2 are coprime, then it is an easy exercise to show that $S(f_1, f_2)$ has a Gröbner representation with respect to $\{f_1, f_2\}$ (this is the so-called "product criterion").

We recall now a fundamental Buchberger criterion for Gröbner bases. For a proof see for example Eisenbud (1995) Lemma 15.1, 15.1 bis and Theorem 15.8. The arguments are given there with Y a finite set, but in fact they depend on the assumption that $K[Y]$ is endowed with a term-ordering. See also the comprehensive Bergman paper (1978) where the theory of Gröbner bases (he didn't use this name) is provided for both commutative and non-commutative setting and without any restriction on the finiteness of the variables set.

Proposition 3.4. *Let G be a basis of an ideal $J \subset K[Y]$. Then G is a Gröbner basis of J if and only if for all $f, g \in G \setminus \{0\}, f \neq g$, the S-polynomial $S(f, g)$ has a Gröbner representation with respect to G.*

This criterion implies a "critical pair & completion" algorithm, transforming a generating set G_0 into a Gröbner basis G. This procedure goes back to Buchberger (1970).

Remark 3.5. If the set Y is infinite then the ring $K[Y]$ is not noetherian and hence it is not guaranteed that G_0, G are finite sets, that is the procedure terminates in a finite number of steps. Of course if the ideal J has a finite basis G_0, then its Gröbner basis G is contained in $K[Y']$, where Y' is the set of variables occurring in G_0. Therefore G is also finite by noetherianity of $K[Y']$. We have a similar situation when J is graded and it has a finite number of generators of degree $\leq d$. Then, there are a finite number of elements of the Gröbner basis of J that have degree $\leq d$, that is the truncated algorithm terminates up to degree d.

When there is the action of a monoid S over a polynomial ring, one has the useful concepts of the S-basis and Gröbner S-basis (see Drensky and La Scala (2006)). We introduce here these notions for the action of $\mathbb{N}$ over $K[X|P]$ by shifting.

Definition 3.6. Let J be an ideal of $K[X|P]$ and $H \subset J$. Then H is said to be a *(Gröbner) shift-basis* of J if $\bigcup_{s \in \mathbb{N}} s \cdot H$ is a (Gröbner) basis of J.

It is clear that if J has a shift-basis then $s \cdot J \subset J$ for all $s \in \mathbb{N}$. Note also that when J is a letterplace ideal, then any letterplace basis of J is a shift-basis but not generally a Gröbner shift-basis of J.

Algorithm 2 GBasis

Input: G_0, a basis of an ideal $J \subset K[Y]$.
Output: G, a Gröbner basis of J.
$G := G_0 \setminus \{0\}$;
$P := \{(f, g) \mid f, g \in G, f \neq g, \gcd(\mathrm{lm}(f), \mathrm{lm}(g)) \neq 1\}$;
while $P \neq \emptyset$ **do**
  choose $(f, g) \in P$;
  $P := P \setminus \{(f, g)\}$;
  $h := \mathrm{Reduce}(S(f, g), G)$;
  if $h \neq 0$ **then**
    $P := P \cup \{(h, g) \mid g \in G, \gcd(\mathrm{lm}(h), \mathrm{lm}(g)) \neq 1\}$;
    $G := G \cup \{h\}$;
  end if;
end while;
return G.

Corollary 3.7. *Let $J \subset K[X|P]$ be a shift-invariant ideal. Then $J^{(0)}$ is a Gröbner shift-basis of the ideal J.*

Proof. By definition $J = \sum_s J^{(s)}$ and $J^{(s)} = s \cdot J^{(0)}$. This implies that $J^{(0)}$ is a shift-basis of J. Let $f \in J^{(u)} \setminus \{0\}, g \in J^{(v)} \setminus \{0\}, f \neq g$ and form the S-polynomial $S(f, g) = cmf - dng$ where $c, d \in K$ and m, n are monomials such that $\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g)) = m\,\mathrm{lm}(f) = n\,\mathrm{lm}(g)$. To prove that $\bigcup_s J^{(s)}$ is a Gröbner basis it is sufficient to show that $S(f, g)$ is shift-uniform, that is $S(f, g) \in \bigcup_s J^{(s)}$. If $u = v$ this is trivial. Assume $u < v$. The variables of m come from the leading monomial of g which has shift v. Then, one has that cmf is shift-uniform with shift u. The variables of n are from the leading monomial of f and one of them has shift u because $u < v$, that is no variable of the leading term of g has shift u. Then also dng is shift-uniform with shift u and the same clearly holds for $S(f, g) = cmf - dng$. □

It is important now to give some condition of compatibility of the term-ordering of $K[X|P]$ with the shift action.

Definition 3.8. A term-ordering on $K[X|P]$ is called *shift-invariant* when $u \prec v$ if and only if $s \cdot u \prec s \cdot v$ for any $u, v \in [X|P]$ and $s \in \mathbb{N}$. In this case one has that $\mathrm{lm}(s \cdot f) = s \cdot \mathrm{lm}(f)$ for all $f \in K[X|P] \setminus \{0\}$ and $s \in \mathbb{N}$.

It is clear that many of the usual term-orderings are shift-invariant. From now on, we assume $K[X|P]$ endowed with a shift-invariant term-ordering.

Proposition 3.9. *Let $J \subset K[X|P]$ be an ideal and $H \subset J$. One has that H is a Gröbner shift-basis of J if and only if $\mathrm{lm}(H)$ is a shift-basis of $\mathrm{LM}(J)$.*

Proof. It is sufficient to note that $\mathrm{lm}(s \cdot H) = s \cdot \mathrm{lm}(H)$ for any $s \in \mathbb{N}$. □

Lemma 3.10. *Let $f_1, f_2 \in K[X|P] \setminus \{0\}, f_1 \neq f_2$. Then $S(s \cdot f_1, s \cdot f_2) = s \cdot S(f_1, f_2)$.*

Proof. We have $\mathrm{lm}(s \cdot f_i) = s \cdot m_i$ where $m_i = \mathrm{lm}(f_i)$ and therefore $\mathrm{lc}(s \cdot f_i) = c_i$ with $c_i = \mathrm{lc}(f_i)$. Put $m = \mathrm{lcm}(m_1, m_2)$ and therefore $s \cdot m = \mathrm{lcm}(s \cdot m_1, s \cdot m_2)$. Then

$$S(s \cdot f_1, s \cdot f_2) = \frac{s \cdot m}{c_1(s \cdot m_1)} s \cdot f_1 - \frac{s \cdot m}{c_2(s \cdot m_2)} s \cdot f_2 = s \cdot S(f_1, f_2). \quad \square$$

By means of the compatibility of the term-ordering of $K[X|P]$ with the action by algebra endomorphisms defined by $\mathbb{N}$, we obtain an example of a Buchberger criterion reduced up to such symmetry.

Proposition 3.11. *Let H be a shift-basis of an ideal $J \subset K[X|P]$. Then H is a Gröbner shift-basis of J if and only if for all $f, g \in H \setminus \{0\}, s \in \mathbb{N}, f \neq s \cdot g$, the S-polynomial $S(f, s \cdot g)$ has a Gröbner representation with respect to $\bigcup_{t \in \mathbb{N}} t \cdot H$.*

Proof. The necessary condition follows from Proposition 3.4. We have to prove now that $G = \bigcup_s s \cdot H$ is a Gröbner basis of J, that is for any $f, g \in H \setminus \{0\}$, $s, t \in \mathbb{N}$, $s \cdot f \neq t \cdot g$ the S-polynomial $S(s \cdot f, t \cdot g)$ has a Gröbner representation with respect to G. Assume $s \leq t$ and put $u = t - s$. By the previous lemma we have $S(s \cdot f, t \cdot g) = S(s \cdot f, s \cdot (u \cdot g)) = s \cdot S(f, u \cdot g)$. By hypothesis, the S-polynomial $S = S(f, u \cdot g)$ is zero or $S = \sum_i f_i g_i$, where $f_i \in K[X|P]$, $g_i \in G$ and $\mathrm{lm}(S) \succeq \mathrm{lm}(f_i)\,\mathrm{lm}(g_i)$ for all i such that $f_i g_i \neq 0$. By acting with the shift s it is clear that $s \cdot S$ also has a Gröbner representation with respect to G. □

By Proposition 3.11 we obtain the correctness of the following algorithm.

Algorithm 3 SGBasis

Input: H_0, a shift-basis of an ideal $J \subset K[X|P]$.
Output: H, a Gröbner shift-basis of J.
$H := H_0 \setminus \{0\}$;
$P := \{(f, s \cdot g) \mid f, g \in H, s \in \mathbb{N}, f \neq s \cdot g, \gcd(\mathrm{lm}(f), \mathrm{lm}(s \cdot g)) \neq 1\}$;
while $P \neq \emptyset$ **do**
  choose $(f, s \cdot g) \in P$;
  $P := P \setminus \{(f, s \cdot g)\}$;
  $h := \mathrm{Reduce}(S(f, s \cdot g), \bigcup_t t \cdot H)$;
  if $h \neq 0$ **then**
    $P := P \cup \{(h, s \cdot g) \mid g \in H, s \in \mathbb{N}, \gcd(\mathrm{lm}(h), \mathrm{lm}(s \cdot g)) \neq 1\}$;
    $P := P \cup \{(g, s \cdot h) \mid g \in H, s \in \mathbb{N}, \gcd(\mathrm{lm}(g), \mathrm{lm}(s \cdot h)) \neq 1\}$;
    $H := H \cup \{h\}$;
  end if;
end while;
return H.

Note that again the termination of this procedure is not guaranteed in general since $X \times P$ and $\bigcup_{s \in \mathbb{N}} s \cdot H_0$ are infinite sets (cf. Remark 3.5 for the situations where the algorithm terminates).

We want now to understand what happens when we apply this algorithm to letterplace ideals of $K[X|P]$. Let $\nu = (\nu_k)_{k \in \mathbb{N}}$ be a multidegree. Define $\sqrt{\nu} = (\eta_k)_{k \in \mathbb{N}}$ where $\eta_k = 1$ if $\nu_k > 0$ and $\eta_k = 0$ otherwise. Moreover, we define

$$V' = \bigoplus_{\sqrt{\nu} = 1^n,\, n \in \mathbb{N}} K[X|P]_{*,\nu} \subset K[X|P]^{(0)}.$$

Lemma 3.12. *Let $f_1, f_2 \in K[X|P] \setminus \{0\}, f_1 \neq f_2$ be homogeneous elements with respect to place-multidegree and consider the S-polynomial $S = S(f_1, f_2) \neq 0$. Assume $\sqrt{\partial_P(f_i)} = s_i \cdot 1^{d_i}$ ($s_i, d_i \in \mathbb{N}$) and $\gcd(\mathrm{lm}(f_1), \mathrm{lm}(f_2)) \neq 1$. Then S is homogeneous with respect to place-multidegree and $\sqrt{\partial_P(S)}$ also has the form $s \cdot 1^d$ with $s, d \in \mathbb{N}$.*

Proof. Put $m_i = \mathrm{lm}(f_i)$ and define $m = \mathrm{lcm}(m_1, m_2)$. Clearly one has that S is homogeneous and $\partial_P(S) = \partial_P(m)$. From $\gcd(m_1, m_2) \neq 1$ it follows that the intersection $\{s_1, \ldots, s_1 + d_1 - 1\} \cap \{s_2, \ldots, s_2 + d_2 - 1\}$ is non-empty and then

$$\{s_1, \ldots, s_1 + d_1 - 1\} \cap \{s_2, \ldots, s_2 + d_2 - 1\} = \{s, \ldots, s + d - 1\}$$

for some s, d. In other words one has that $\sqrt{\partial_P(m)} = s \cdot 1^d$. □

Proposition 3.13. *Let $J \subset K[X|P]$ be a letterplace ideal. There exists a Gröbner shift-basis of J contained in $\bigcup_\nu J_{*,\nu} \cap V'$.*

Proof. We argue by induction on the countable steps of the algorithm SGBasis when applied to a letterplace basis $H_0 \subset \bigcup_d J_d \cap V$ of the ideal J. Note that any element $f \in H_0$ is homogeneous with respect to place-multidegree and $\partial_P(f) = 1^d = \sqrt{\partial_P(f)}$ for some $d \in \mathbb{N}$. Suppose now that at the current step, one has to reduce the S-polynomial $S = S(f, s \cdot g)$ of the elements $f, g \in H$ where $f \neq s \cdot g$ and $\gcd(\mathrm{lm}(f), \mathrm{lm}(s \cdot g)) \neq 1$. By induction, we can assume $\sqrt{\partial_P(f)} = 1^{d'}$, $\sqrt{\partial_P(g)} = 1^{d''}$

and therefore $\sqrt{\partial_P(S)} = 1^d$ for some d, by the previous lemma. Since the procedure Reduce clearly preserves place-multigrading, one has that the remainder $h = \text{Reduce}(S(f, s\cdot g), \bigcup_t t\cdot H)$ is either zero or satisfies $\sqrt{\partial_P(h)} = 1^d$. We conclude that a Gröbner shift-basis $H \subset J$ is contained in $\bigcup_\nu J_{*,\nu} \cap V'$. □

Definition 3.14. Let J be a letterplace ideal of $K[X|P]$ and $H \subset J$. We say that H is a *Gröbner letterplace basis* of J if $H \subset \bigcup_\nu J_{*,\nu} \cap V'$ and H is a Gröbner shift-basis of J.

To obtain a homogeneous Gröbner basis of a graded two-sided ideal I from a Gröbner letterplace basis of the letterplace analogue $J = \tilde{\iota}(I)$, one has to provide the following compatibility condition of the term-orderings with the map ι.

Definition 3.15. Fix the term-orderings $<$ on $K\langle X\rangle$ and $\prec$ on $K[X|P]$. They are called *compatible with* ι, when $v < w$ holds if and only if $\iota(v) \prec \iota(w)$ for any $v, w \in \langle X\rangle$. In this case it follows that $\mathrm{lm}(\iota(f)) = \iota(\mathrm{lm}(f))$ for all $f \in K\langle X\rangle \setminus \{0\}$.

There are many term-orderings that are compatible with ι. For instance, order the set X in the natural way ($x_i < x_k$ when $i < k$) and the set $X \times P$ by putting $(x_i|j) \prec (x_k|l)$ if $i + j < k + l$ or $i+j = k+l$, $i < k$. Define also on $X \times P$ the ordering $(x_i|j) < (x_k|l)$ if $j < l$ or $j = l$, $i < k$. Use the latter order to define a canonical form for the monomials $m \in [X|P]$, that is $m = (x_{i_1}|j_1)\cdots(x_{i_n}|j_n)$ where $(x_{i_1}|j_1) \le \cdots \le (x_{i_n}|j_n)$ (and hence $j_1 \le \cdots \le j_n$). Now refine the partial ordering on the degrees of the monomials in $[X|P]$ in the following way. Let $m = (x_{i_1}|j_1)\cdots(x_{i_n}|j_n)$, $m' = (x_{k_1}|l_1)\cdots(x_{k_n}|l_n)$ be two monomials of the same degree. We define $m \prec m'$ when $(x_{i_p}|j_p) = (x_{k_p}|l_p)$ for $p = 1, 2, \ldots, q-1$ and $(x_{i_q}|j_q) \prec (x_{k_q}|l_q)$ $(1 \le q \le n)$. It is clear that $\prec$ is a term-ordering on $K[X|P]$. We fix now on $K\langle X\rangle$ the length lexicographic order that we will denote as $<$. When $m, m' \in V$ (that is $j_1 = l_1 = 0, \ldots, j_n = l_n = n-1$) one has that $m \prec m'$ if and only if $\iota^{-1}(m) = x_{i_1}\cdots x_{i_n} < x_{k_1}\cdots x_{k_n} = \iota^{-1}(m')$, that is $<, \prec$ are ι-compatible. Moreover, the term-ordering $\prec$ is clearly shift-invariant.

In what follows we assume that the algebras $K\langle X\rangle, K[X|P]$ are endowed with term-orderings compatible with ι, where the one of $K[X|P]$ is also shift-invariant.

Proposition 3.16. *Let $I \subset K\langle X\rangle$ be a graded two-sided ideal and put $J = \tilde{\iota}(I)$. Moreover, let H be a Gröbner letterplace basis of J and put $G = \iota^{-1}(H \cap V) \subset \bigcup_{d\in\mathbb{N}} I_d$. Then G is a Gröbner basis of I as two-sided ideal.*

Proof. Let $f \in I_d$ and put $f' = \iota(f)$. Then, there is $m \in [X|P], s \in \mathbb{N}, h \in H$ such that $\mathrm{lm}(f') = m\,\mathrm{lm}(s\cdot h) = m(s\cdot \mathrm{lm}(h))$. From $f' \in J_d \cap V$ and $\sqrt{\partial_P(h)} = 1^n$ $(n \in \mathbb{N})$ it follows necessarily that $\partial_P(h) = 1^n$, that is $h \in H \cap V$. This implies that $\mathrm{lm}(f') = m(s\cdot \mathrm{lm}(h)) = m_1(s\cdot \mathrm{lm}(h))((s+n)\cdot m_2)$ where $m_1, m_2 \in [X|P] \cap V$ and $s = \deg(m_1)$. Since the term-orderings are compatible with ι, we obtain that $\mathrm{lm}(f) = w_1\,\mathrm{lm}(g) w_2$ where $g = \iota^{-1}(h)$, $w_i = \iota^{-1}(m_i)$. □

Lemma 3.17. *Let $f_1, f_2 \in K[X|P] \setminus \{0\}, f_1 \neq f_2$ be homogeneous elements with respect to place-multidegree and consider the S-polynomial $S = S(f_1, f_2) \neq 0$. Assume $\sqrt{\partial_P(f_i)} = s_i \cdot 1^{d_i}$ $(s_i, d_i \in \mathbb{N})$ and $\gcd(\mathrm{lm}(f_1), \mathrm{lm}(f_2)) \neq 1$. If $\partial_P(S) = s\cdot 1^d$ $(s, d \in \mathbb{N})$ then $\partial_P(f_i) = s_i \cdot 1^{d_i}$ $(i = 1, 2)$.*

Proof. Put $m_i = \mathrm{lm}(f_i)$ and define $m = \mathrm{lcm}(m_1, m_2)$. If $m = (x_{i_1}|j_1)\cdots(x_{i_d}|j_d)$ with $j_1 < \cdots < j_d$ then the same happens also for m_1 and m_2. □

Let $I \subset K\langle X\rangle$ be a graded two-sided ideal generated by $G_0 \subset \bigcup_{d\in\mathbb{N}} I_d$. Owing to Proposition 3.16, to compute a homogeneous Gröbner basis G of I one can compute the Gröbner letterplace basis H of $J = \tilde{\iota}(I)$ by applying the algorithm SGBasis to $H_0 = \iota(G_0)$. Actually $G = \iota^{-1}(H \cap V)$ and hence we are interested in calculating just the elements of $H \cap V$. Lemma 3.17 implies that they are all obtained by reducing S-polynomials $S(f, s\cdot g)$ where f, g are already elements in V. In other words, we have the correctness of Algorithm 4 that we propose as an alternative method for computing non-commutative homogeneous Gröbner bases.

Clearly the termination of this procedure is not provided in general, even if the set of variables X is finite and the ideal I has a finite basis G_0. From the viewpoint of our method, this corresponds to the fact that the set of letterplace variables $X \times P$ is infinite, and the letterplace ideal $J = \tilde{\iota}(I)$ is generated by $\bigcup_{s\in\mathbb{N}} s\cdot\iota(G_0)$ which is also an infinite set. Nevertheless, we have the following result concerning termination of the truncated version of the algorithm.

Algorithm 4 NCGBasis

Input: G_0, a homogeneous basis of a graded two-sided ideal $I \subset K\langle X\rangle$.
Output: G, a homogeneous Gröbner basis of I as two-sided ideal.
$H := \iota(G_0 \setminus \{0\})$;
$P := \{(f, s\cdot g) \mid f, g \in H, s \in \mathbb{N}, f \neq s\cdot g, \gcd(\mathrm{lm}(f), \mathrm{lm}(s\cdot g)) \neq 1,$
$\quad\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(s\cdot g)) \in V\}$;
while $P \neq \emptyset$ **do**
  choose $(f, s\cdot g) \in P$;
  $P := P \setminus \{(f, s\cdot g)\}$;
  $h := \textsc{Reduce}(S(f, s\cdot g), \bigcup_t t\cdot H)$;
  if $h \neq 0$ **then**
    $P := P \cup \{(h, s\cdot g) \mid g \in H, s \in \mathbb{N}, \gcd(\mathrm{lm}(h), \mathrm{lm}(s\cdot g)) \neq 1,$
    $\quad\mathrm{lcm}(\mathrm{lm}(h), \mathrm{lm}(s\cdot g)) \in V\}$;
    $P := P \cup \{(g, s\cdot h) \mid g \in H, s \in \mathbb{N}, \gcd(\mathrm{lm}(g), \mathrm{lm}(s\cdot h)) \neq 1,$
    $\quad\mathrm{lcm}(\mathrm{lm}(g), \mathrm{lm}(s\cdot h)) \in V\}$;
    $H := H \cup \{h\}$;
  end if;
end while;
$G := \iota^{-1}(H)$;
return G.

Proposition 3.18. *Let $I \subset K\langle X\rangle$ be a graded two-sided ideal and $d > 0$ an integer. If I has a finite number of homogeneous generators of degree $\leq d$ then the algorithm* NCGBasis *computes in a finite number of steps all elements of degree $\leq d$ of a homogeneous Gröbner basis of I.*

Proof. With the notation of the algorithm NCGBasis, consider the elements $f, g \in H \subset V$ at the current step. If both of these polynomials have degree $\leq d$ then the condition $\gcd(\mathrm{lm}(h), \mathrm{lm}(s\cdot g)) \neq 1$ implies that $s \leq d - 1$. It follows that the computation actually runs over the variables set $X' \times \{0, \ldots, d-1\}$, where X' is the finite set of variables occurring in the generators of I of degree $\leq d$. By noetherianity of the ring $K[X' \times \{0, \ldots, d-1\}]$ we conclude that the truncated procedure, up to degree d, stops after a finite number of steps. □

Corollary 3.19. *Let $I \subset K\langle X\rangle$ be a graded two-sided ideal and assume that it has a finite homogeneous basis whose polynomials are all of degree $\leq d$. Denote by G_{d-1} the output of* NCGBasis *up to degree d. If $G_{d-1} = G_{2d-2}$ then G_{d-1} is a Gröbner basis of I.*

Proof. Put $H_{d-1} = \iota(G_{d-1})$. If $H_{d-1} = H_{2d-2}$, this means that all the S-polynomials $S(f, s\cdot g)$ with $f, g \in H_{d-1}, s \in \mathbb{N}, f \neq s\cdot g, \gcd(\mathrm{lm}(f), \mathrm{lm}(s\cdot g)) \neq 1$ (and hence $s \leq 2d-2$) and $\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(s\cdot g)) \in V$ reduce to zero with respect to $\bigcup_t t\cdot H_{d-1}$. Then, by Proposition 3.11 and Proposition 3.16 one obtains the claim. □

Note that the Proposition 3.18 generalizes a well-known result concerning solvability of word problems for finitely presented homogeneous associative algebras. Moreover, the algorithm provided by this proposition is important for concrete applications, since even if an ideal has an infinite Gröbner basis this need not be unpleasant. A partial knowledge of such basis may be enough for predicting formulas that can be applied to determine various invariants (see Drensky and La Scala (2006) and Ufnarovski (1989)). In general, it is still an open problem to determine when a finitely generated ideal has also a finite Gröbner basis with respect to a given monomial ordering. We want also to mention that the assumption about homogeneity of two-sided ideals needed by the proposed method is not too restrictive. In fact, it is well-known that a Gröbner basis of any two-sided ideal I can be obtained via a Gröbner basis of a homogenized version of I (see for instance Nordbeck (1998)). Nevertheless, we hope in the future to have an extension of the ideal correspondence $\tilde{\iota}$ and related algorithms for non-graded ideals. The feasibility of the algorithm NCGBasis is tested in Section 5 where we compare an experimental implementation that we developed in the computer algebra system Singular (Greuel et al., 2006) with some of the best implementations of classical algorithms for non-commutative Gröbner bases.

4. A concrete example

In this section, we compute a non-trivial example with the classical non-commutative Buchberger algorithm as well as with Algorithm 4.

Let $X = \{x, y\}$. Consider $f_1 = x^3 - y^3 = xxx - yyy$, $f_2 = xyx - yxy$ and $I = \langle f_1, f_2 \rangle \subset K\langle X\rangle$ with respect to the graded left lexicographical ordering. We compute the truncated Gröbner basis up to degree $d = 5$.

4.1. Computation in $K\langle X\rangle$

We follow the common terminology and refer to an s-polynomial in this setting rather as an *overlap*. Having two words $w = ow'$ and $v = v'o$, we denote, for short, their overlap at o by $v' \cdot o \cdot w'$ below.

Let $G = \{f_1, f_2\}$ as above.

(f_1, f_1) : $\mathrm{lm}(f_1) = xxx$, so there are two self-overlaps of f_1, namely

$$o_1 := o_{1,1} = f_1x - xf_1 = xy^3 - y^3x, \qquad o_{1,2} = f_1x^2 - x^2f_1 = x^2y^3 - y^3x^2.$$

Moreover, $o_{1,2} - xo_{1,1} = xy^3x - y^3x^2 = o_{1,1}x$, so $o_{1,2}$ reduces to 0. Hence $G = G \cup \{o_1\}$.

(f_2, f_2) : $\mathrm{lm}(f_2) = xyx$, so there are two self-overlaps. Indeed, due to symmetry, these two are redundant, since they originate from the overlap $xy \cdot x \cdot yx$ of $\mathrm{lm}(f_2)$. Then

$$o_2 = f_2yx - xyf_2 = xyyxy - yxyyx. \text{ So } G = G \cup \{o_2\}.$$

(f_1, f_2) : $\mathrm{lm}(f_1)$ and $\mathrm{lm}(f_2)$ have two overlaps $xx \cdot x \cdot yx$ and $xy \cdot x \cdot xx$; hence

$$o_{3,1} = f_1yx - xxf_2 = xxyxy - y^4x \quad \text{and} \quad o_{3,2} = f_2xx - xyf_1 = xy^4 - yxyxx.$$

Performing reductions, we see that $o_{3,1} - xf_2y - f_2yy - yo_1 = 0$ and $o_{32} - o_1y + yf_2x + yyf_2 = yyyxy - yyyxy = 0$.

(f_1, o_1) has overlap $xx \cdot x \cdot yyy$, (f_2, o_1) has overlap $xy \cdot x \cdot yyy$, (f_1, o_2) has overlap $xx \cdot x \cdot yyxy$, (o_1, o_2) has overlap $xyy \cdot xy \cdot yy$, o_2 has a self-overlap $xyy \cdot xy \cdot yxy$ and (f_2, o_2) has two overlaps $xy \cdot x \cdot yyxy$ and $xyy \cdot xy \cdot x$. Since all these elements are of degree ≥ 6 and we are in the graded case, we conclude that $G = \{f_1, f_2, o_1, o_2\}$ is the truncated Gröbner basis up to degree 5.

4.2. Computation in $K[X|P]$

Assume for simplicity that the lowest shift is 1 (instead of 0) and let us denote the variables $(x|i)$, $(y|j)$ as $x(i), y(i)$. For the case $d = 5$, one considers then the polynomial ring $K[X|P_5] = K[x(1), y(1), \ldots, x(5), y(5)]$ and the polynomials

$f_1 = x(1)x(2)x(3) - y(1)y(2)y(3)$,
$f_2 = x(2)x(3)x(4) - y(2)y(3)y(4) = 1 \cdot f_1$,
$f_3 = x(3)x(4)x(5) - y(3)y(4)y(5) = 2 \cdot f_1$,
$f_4 = x(1)y(2)x(3) - y(1)x(2)y(3)$,
$f_5 = x(2)y(3)x(4) - y(2)x(3)y(4) = 1 \cdot f_4$,
$f_6 = x(3)y(4)x(5) - y(3)x(4)y(5) = 2 \cdot f_4$.

Out of set P of 15 pairs (f_i, f_j), which we write for short as just (i, j),

- the pairs $(1, 4)$, $(1, 5)$, $(2, 4)$, $(4, 5)$ are discarded by the V-criterion, since the lcm of leading monomials is non-multilinear in places; note that $(4, 5)$ can also be discarded by the product criterion;
- the pairs $(2, 3)$, $(2, 5)$, $(2, 6)$, $(3, 5)$, $(3, 6)$, $(5, 6)$ are discarded by the V-criterion, since the lcm of leading monomials has non-zero shift; note that $(5, 6)$ can also be discarded by the product criterion.

Hence, it remains to consider five pairs $P = \{(1, 2), (1, 3), (1, 6), (3, 4), (4, 6)\}$. Following Algorithm 4, $H := \{f_1, f_4\}$.

$\mathrm{spoly}(1, 2) = f_1x(4) - x(1)f_2 = x(1)y(2)y(3)y(4) - y(1)y(2)y(3)x(4) =: g_1$; hence $H := H \cup \{g_1\}$ and we denote by $g_2 := 1 \cdot \mathrm{spoly}(1, 2) = x(2)y(3)y(4)y(5) - y(2)y(3)y(4)x(5)$ the only admissible shift of g_1.

$\mathrm{spoly}(1, 3) = f_1x(4)x(5) - x(1)x(2)f_3 = x(1)x(2)y(3)y(4)y(5) - y(1)y(2)y(3)x(4)x(5) = x(1)g_2 + g_1x(5) \to 0$. Note that we can apply the chain criterion to pairs $(1, 2)$, $(1, 3)$, $(2, 3)$ and since

$\mathrm{lm}(f_2) \mid \mathrm{lcm}(\mathrm{lm}(f_1), \mathrm{lm}(f_3))$ we can skip the pair $(1, 3)$. The pair $(2, 3)$ is skipped by the V-criterion above.
$\mathrm{spoly}(1, 6) = f_1y(4)x(5) - x(1)x(2)f_6 = x(1)x(2)y(3)x(4)x(5) - x(1)x(2)x(3)y(4)x(5)$. Indeed, $\mathrm{spoly}(1, 6) = x(1)f_5y(5) + f_4y(4)y(5) + y(1)g_2 \to 0$.
$\mathrm{spoly}(3, 4) = f_4x(4)x(5) - x(1)y(2)f_3 = x(1)y(2)y(3)y(4)y(5) - y(1)x(2)y(3)x(4)x(5)$ and $\mathrm{spoly}(3, 4) = g_1y(5) - y(1)f_5x(5) - y(1)y(2)f_6 \to 0$.
$\mathrm{spoly}(4, 6) = f_4y(4)x(5) - x(1)y(2)f_6 = x(1)y(2)y(3)x(4)y(5) - y(1)x(2)y(3)y(4)x(5)$ cannot be reduced; hence $g_3 := \mathrm{spoly}(4, 6)$ and H becomes $\{f_1, f_4, g_1, g_3\}$. Note that there are no admissible shifts for g_3.

Inspection of the pairs $(g_1, f_1), \ldots, (g_1, f_6), (f_1, g_2), (f_4, g_2)$, appearing when g_1 enters H and $(g_3, f_1), \ldots, (g_3, f_6), (g_3, g_1), (g_3, g_2)$, which appear when g_3 enters H (note that the pairs $(f_1, g_3), (f_4, g_3), (g_1, g_3)$ are already included in the latter set) shows that all of these pairs are discarded by the V-criterion.

Thus $\iota^{-1}(\{f_1, f_4, g_1, g_3\})$ is a truncated Gröbner basis up to degree 5 of $I \subset K\langle X\rangle$.

5. Implementation and comparison

Among the computer algebra systems, there are only a few which provide a user with the possibility of performing computations in free associative algebras and path algebras. The following is an exhaustive list, to the best of our knowledge, of such systems.

- Bergman, from Backelin et al. (2006), is a powerful and flexible tool for calculating Gröbner bases, Hilbert and Poincaré–Betti series, Anick resolutions, and Betti numbers in non-commutative algebras and in modules over them. Per default, Bergman takes homogeneous polynomials as the input. However, one is able to compute Gröbner bases of non-homogeneous ideals using homogenization and the so-called *rabbit strategy* provided by Bergman.
- NCGB, from Helton and Stankus (2001), is a package for Mathematica, partially written in C. It is a part of the NCAlgebra suite, which performs various operations (e.g. simplification and reduction modulo a Gröbner basis) of non-commutative expressions.
- Opal, from Green et al. (1997), is the specialized standalone system for Gröbner bases in free and path algebras. Opal does not require the homogeneity of an input and is able to compute degree-bounded Gröbner basis.
- GBNP (also called Grobner), from Cohen and Gijsbers (2003), is a package for Gap 4 with the implementation of non-commutative Gröbner bases for free and path algebras, following the algorithmic approach of Mora (1989, 1994). It is a recent development, gaining more and more functionality with every new release.
- Felix, from Apel and Klaus (1998), provides generalizations of Buchberger's algorithm to free K-algebras, polynomial rings and non-commutative G-algebras. Also, syzygy computations and basic ideal operations are implemented.
- in Magma, from Bosma et al. (1997), there is, among others, a generalization of Buchberger's algorithm to one-sided and two-sided ideals of finitely presented K-algebras as well as a non-commutative generalization (due to Allan Steel) of the Faugère F4 algorithm. These developments are quite recent in Magma. There are basic ideal operations and very important vector enumeration tools implemented.

Our experimental implementation of the algorithm NCGBasis consists of two parts: the kernel part, realized in the kernel of Singular, and the interface part, containing additional useful procedures (such as conversion between different presentations of objects), which are collected in the Singular library `freegb.lib`. We would like to stress the fact that in the kernel part, we were basing consideration on the most common Singular internal routines for Buchberger's algorithm. That is, we were not using any other variants of the algorithm like F_4-based or F_5-based, Hilbert or syzygy-driven algorithms and so on. There is still a lot of room for improvements in our prototype implementation.

The original ordering on the variables of the free algebra $K\langle X\rangle = K\langle x_0, \ldots, x_{n-1}\rangle$ is expanded blockwise to the ordering on the letterplace polynomial ring $K[X|P_d]$ where the places are bound by the degree d that one wants to reach, that is

$$K[X|P_d] = K[(x_0|0), \ldots, (x_{n-1}|0), \ldots, (x_0|d-1), \ldots, (x_{n-1}|d-1)].$$

Thus, the latter ordering consists of d blocks with the original ordering. Note that in the Singular interface, a letterplace variable $(x_{i-1}|j-1)$ is written as $x_i(j)$ in order to have a compact format and avoid zero indices and places. By following this convention, we provide users with the possibility of performing computations with respect to the variety of term-orderings available in Singular and compatible with ι.

The tests were performed on a PC equipped with two Intel Pentium4 Processors 3200 MHz with 4 GB RAM running Linux. However, it was possible to use only one processor and at most 2GB RAM for our processes. To compare our implementation of NCGBasis with classical algorithms for non-commutative Gröbner bases we have used:

- Bergman version 1.01,
- GBNP release 0.9.3 on GAP 4 release 4,
- OPAL version 1.0,
- Singular 3-0-4 with `freegb.lib` version 1.9,
- Magma version 2.14-16.

Up to now, there is still no publicly available collection of standard benchmarks for non-commutative Gröbner bases in free and path algebras. Good sources of examples are Keller (1997) and Ufnarovski (1995) and the GBNP user manual (Cohen and Gijsbers, 2003). However, these examples are too easy for modern systems. We used the ideas, collected from the above references and composed our own examples, partially based on them. Next, we went through the algebraic literature and picked up more relevant examples, which we describe below. We are grateful to Victor Ufnarovski for sending us some very interesting examples.

We would like to stress the importance of creating a unified set of examples, which will serve as benchmarks for systems, computing non-commutative Gröbner bases in free and path algebras. It seems possible to use the tools developed in the SymbolicData project to organize the systematic collection and the work with the representative examples.

Out of many examples, we took the most interesting ones, such as examples with Gröbner bases containing elements of high degree or the infinite Gröbner basis. We selected those examples for the presentation whose running time, obtained with the fastest system, ranges from 1 s to 4 min.

Now we describe the examples which have been chosen for testing.

Example 5.1. Consider the two-sided ideal I, such that $K\langle X\rangle/I$ is the universal enveloping algebra of the (relatively) free nilpotent Lie algebra L of class c. In other words, the ideal I is generated by all (left-normed) commutators $[x_{i_1}, \ldots, x_{i_{c+1}}]$ of length $c+1$, where the number of variables $x_i \in X$ is the dimension n of the algebra L. In particular, we study the case when $n = 5$ and $c = 3, 4$. We called these examples `nilp3` and `nilp4`. We compute up to degrees 6, 10 for `nilp3` and 6, 7, 8 for `nilp4`. We use the example `nilp4` in two forms, namely with the big list of totally 2500 redundant relations (`nilp4`) obtained by an automatic generation procedure, and the list of 1200 simplified relations (`nilp4s`).

Example 5.2. Another example of the same kind is the ideal I, defining the universal enveloping algebra of the free metabelian Lie algebra. In other words, I is generated by all the commutators $[[x_i, x_j], [x_k, x_l]]$. We fix the dimension to be $n = 5$, denote this example as `metab5` and compute up to degrees 10 and 11. There are 360 generators, while in the examples denoted by `metab5s` we use 45 irredundant ones.

In the theory of associative algebras a fundamental role is played by the so-called *T-ideals* which are (multi)graded two-sided ideals I of the free associative algebra $K\langle X\rangle$ given by all polynomials which are zero when evaluated on elements of an algebra A. Then, A is said a *PI-algebra* if I is different from zero. Usually the T-ideals are not finitely generated as ideals of $K\langle X\rangle$, and so one can give a finite set of generators just up to some degree d.

Example 5.3. As an example for testing our implementation we consider the T-ideal I of the algebra of 2-by-2 upper triangular matrices. We have that I is generated by polynomials $[x_i, x_j]w[x_k, x_l]$ where

w is an arbitrary word (including 1) of $K\langle X\rangle$. For the test we fix the number of variables equal to 4 and degree bound to 7, and denote this example as `tri4`. We compute with this generating set of 12 240 polynomials and with the simplified set (in the example `tri4s`) of 3060 generators.

A very rich source of examples is provided by Serre's relations. For a square integer matrix $A = (a_{ij})$, for $i \neq j$ we have $1 - a_{ij} \in \mathbb{Z}_+$, so the (i, j)-th relation reads $\mathrm{ad}_{x_j}^{1-a_{ij}}(x_i) = [x_j, [\ldots[x_j, x_i]]\ldots]$. We consider Cartan matrices, which lead to simple Lie algebras as well as generalized Cartan matrices, which are used to define Kac–Moody algebras. Serre's relations are homogeneous; they appear between the generators of positive or negative parts of an algebra. For both universal enveloping algebras of Lie algebras and Kac–Moody algebras, Serre's relations are very important. Together with additional relations (which are non-homogeneous), Serre's relations lead to compact presentations of algebras. On the way to the general case we work first with the homogeneous relations. Note that in the case of Lie algebras, employing a degree ordering leads to finite Gröbner basis. This is not the case for general Kac–Moody algebras, which makes the latter algebras more interesting to study from the algorithmic point of view.

Example 5.4. The Cartan matrices for the algebras F_4 and E_6 are well-known and can be obtained explicitly with e.g. GAP. The generalized Cartan matrices for HA_1^1 and for the $EHA_1^{1,2}$ (which is an instance of parametric extended HA_1^1 matrix) are the following:

$$HA_1^1 : \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}, \qquad EHA_1^{1,2} : \begin{pmatrix} 2 & -2 & -3 \\ -2 & 2 & -1 \\ -2 & -5 & 2 \end{pmatrix}.$$

We call these examples `ser-f4`, `ser-e6`, `ser-ha` and `ser-eha` respectively.

For the purpose of a fast comparison with many systems, we produce also the following set of examples, which we constructed ourselves or got from other sources. The running time on Singular for such examples is less than half a minute.

Example	Generators of an ideal	#Out
`braid3-11`	$yxy - zyz, xyx - zxy, zxz - yzx, x^3 + y^3 + z^3 + xyz$	726
`braid4-11`	$yxy - zyz, xyz - zxy, zxz - yzx, x^3 + y^3 + z^3 + xyz$	416
`lp1-10`	$z^4 + yxyx - xy^2x - 3zyxz, x^3 + yxy - xyx, zyx - xyz + zxz$	55
`lv2-15`	$xy + yz, x^2 + xy - yx - y^2$	184
`ufn1h-11`	see below	360
`ufn1h-14`	see below	712
`ufn1h-15`	see below	892

The example `ufn1h` is a homogenization of the example, communicated to us by Victor Ufnarovski. The ideal is generated by the set $\{a^2 - ah, b^2 - bh, c^2 - ch, d^2 - dh, aba - abh, bab - abh, aca - ach, cac - ach, ada - adh, dad - adh, bcb - bch, cbc - bch, bdb - bdh, dbd - bdh, cdc - cdh, dcd - cdh, ha - ah, hb - bh, hc - ch, hd - dh\}$ in the algebra $K\langle a, b, c, d, h\rangle$. Note that the original inhomogeneous set of generators has finite Gröbner basis, while homogenizing the generators leads to an infinite Gröbner basis. We consider also another example from Victor Ufnarovski denoted as `ufn3`. This is a list of 125 binomials of degree 2 in 15 variables. Some of them represent anti-commutativity, $ab + ba$; the rest are of the form $ab + cd$, $ca + ab$, $de + fd$ and so on.

In the last column of the table we write the number of computed elements of the Gröbner basis. We denote each example by its name and the degree bound up to which the computation must go; like `lp1-10` means we compute up to degree 10 with the example `lp1`. For all the examples, we fix the term-ordering as the length left lexicographic one and the ground field $K = \mathbb{Q}$.

We measure the total running time of each call to a system in a batch mode. In this time the initialization of a system, loading of an example file, the actual computation and the writing of an output are included. Since Magma has two algorithms for computing non-commutative Gröbner bases, we test both of them and denote them as Magma GB and Magma F_4 respectively. We have to make a remark on GBNP (of GAP) and on Magma. Since GAP and Magma are especially big systems, the loading process of all needed packages took about 3–4 s for GAP and 5–7 s for Magma in our configuration. Since systems like Bergman and Singular load their standard tools too (although the loading takes up to 1–2 s), we do not subtract this time from the total one.

The running times in the tables below are given in "minutes:seconds" format.

Example	Bergman	GBNP	OPAL	Singular	Magma GB	Magma F_4
`braid3-11`	1:11	8:31	$80{:}00^{\dagger}$	0:17	2:26	2:08
`braid4-11`	0:14	1:12	33:07	0:04	0:54	0:53
`lp1-10`	0:07	0:20	11:40	0:01	1:23	1:29
`lv2-15`	0:05	1:17	$98{:}00^{\dagger}$	0:02	0:13	0:13
`ufn1h-11`	0:02	0:14	0:09	0:02	0:17	0:19
`ufn1h-14`	0:09	2:39	0:38	0:13	0:23	0:28
`ufn1h-15`	0:14	10:26	0:57	0:23	0:27	0:26

For the timings we use the following shortcuts. We write $t^{\dagger}$ when the system call exited with non-zero status after the time t. Then, $t^{\dagger\dagger}$ means that the system call exited with non-zero status after the time t, but the dump returned the correct number of generators. Finally, we denote by $t^{\times}$ that the process was terminated after the time t.

From this table, and other experiments, we conclude that, in general, OPAL cannot compete with the other systems on the variety of examples of different natures. Therefore we do not test advanced examples with this system.

In the following table we gather the information on the behaviour of four systems on much harder examples, which were described before. In addition, we print the number of generators in the input and in the output.

Example	Bergman	GBNP	Singular	MagmaGB	MagmaF4	#In	#Out
`nilp3-6`	0:01	0:07	0:01	0:16	0:15	192	110
`nilp3-10`	0:23	1:49	0:03	0:38	1:59	192	110
`nilp4-6`	1:22	1:12	0:14	1:55	1:42	2 500	891
`nilp4-7`	1:24	7:32	1:40	6:48	5:09	2 500	1238
`nilp4s-8`	13:52	74:54	$0{:}57^{\dagger}$	27:29	12:16	1 200	1415
`metab5-10`	0:20	$13{:}58^{\dagger\dagger}$	0:22	3:08	3:16	360	76
`metab5-11`	27:23	$14{:}42^{\dagger}$	1:11	30:43	30:06	360	113
`metab5s-10`	0:32	$102{:}43^{\dagger\dagger}$	0:34	3:23	3:11	45	76
`metab5s-11`	27:33	$25{:}27^{\dagger}$	2:05	30:39	28:04	45	113
`tri4-7`	0:48	$1080{:}00^{\times}$	0:08	0:36	31:58	12 240	672
`tri4s-7`	0:40	3:37	0:07	0:29	0:37	3 060	672
`ser-f4-15`	16:05	85:48	0:08	15:03	1:58	9	43
`ser-e6-12`	0:49	5:39	0:07	0:32	0:37	20	76
`ser-e6-13`	2:36	14:52	0:14	1:29	1:16	20	79
`ser-ha-10`	0:04	7:82	0:01	0:27	0:20	5	33
`ser-ha-15`	63:21	246:00	1:58	21:15	16:45	5	112
`ser-eha-12`	0:56	3:44	0:37	8:08	8:36	6	126
`ser-eha-13`	72:50	34:53	4:08	35:38	35:21	6	174
`ufn3-6`	0:31	1:43	0:23	0:40	0:28	125	1065
`ufn3-8`	2:18	9:33	2:20	1:14	0:44	125	1763
`ufn3-10`	5:24	20:37	$3{:}25^{\dagger}$	1:57	1:04	125	2446

As we can see, for complicated examples our implementation in Singular scores better results than Bergman and both algorithms of Magma, and very often these systems are faster than GBNP. The latter clearly has a problem with handling large inputs, because it launches a minimization or a kind of interreduction first. This approach is known to be counter-productive even in the commutative case. Analyzing the output of the failed examples, we see that after hours of work, GBNP was stuck in the interreduction procedure. When we input the irredundant set of relations for the same ideal, GBNP is able to compute a Gröbner basis.

At the present state of our experimental implementation in SINGULAR, we have experienced one drawback concerning a large use of memory during the computation of some tests with high degrees and many variables. This depends on the fact that we actually store not only new generators, but also their shifted polynomials. Note that this greedy behaviour as regards the memory is not a problem of the theoretical algorithm, but an issue of our prototype implementation only. We are working further to enhance our implementation. In the following table we show the number of times three criteria have been applied for a couple of examples. These numbers are counted internally by SINGULAR and show that due to the V-criterion, that is $\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g)) \in V$, we skip a very large number of pairs, incomparable to the number of times that either product or chain criteria apply.

Example	Product criterion	Chain criterion	V-criterion
braid-3-11	1 039	200	725 307
ufn1h-11	7 824	205	377 438
ufn1h-15	41 900	553	3011 860
serre-eha112-13	402	25	77 587

The timings that we obtained in all the experiments are very encouraging and show that in addition to the interesting feature of being portable to any commutative computer algebra system, the method that we propose is really feasible. Our implementation will become available to the general public together with the new major release 3-1 of SINGULAR, which is about to appear. It also seems possible to us to obtain all non-commutative Gröbner basics such as intersections of ideals, kernels of ring homomorphisms etc. and also to extend the algorithm in order to do the computation of syzygies, resolutions, dimensions and so on. Thus, we may say that this article proposes a new computing paradigm – a letterplace technology – for effective computations with non-commutative algebras (via commutative ones). Another good point is the possibility of using the variety of classical monomial orderings from the commutative case in order to model the orderings on a letterplace algebra. In fact, computer algebra systems for free algebras usually lack a considerable choice of term-orderings.

Acknowledgments

The authors wish to express their gratitude to Bruno Buchberger and Gert-Martin Greuel for their constructive criticisms and suggestions. We are indebted to Hans Schönemann for his cooperation on the implementation of the algorithms. We would also like to thank Victor Ufnarovski for his valuable hints concerning the examples and also for his kind assistance in the use of the system BERGMAN. Anonymous referees expressed valuable remarks and thus deserve special thanks.

The first author was partially supported by Università di Bari, Ministero dell'Università e della Ricerca, and by the DFG Graduiertenkolleg "Hierarchie und Symmetrie in mathematischen Modellen" at RWTH Aachen, Germany.

Appendix. Example scripts, used in tests

As an illustration, we take the `lp1-10` test example.

BERGMAN : `lp1-10.bg`

```
(off gc)
(noncommify)
(setmaxdeg 10)
(simple)
vars z,y,x;
```

```
z^4 + y*x*y*x - x*y*y*x - 3*z*y*x*z, x^3 + y*x*y - x*y*x,
z*y*x-x*y*z + z*x*z;
(quit)
```

Command line call: `bergman` < `lp1-10.bg` > `lp1-10.bg.res`.

GBNP: `lp1-10.gap`

```
LoadPackage("GBNP","0",false);;
SetInfoLevel(InfoGBNP,1);
SetInfoLevel(InfoGBNPTime,1);
K:=Rationals;;
A:=FreeAssociativeAlgebraWithOne(K,"z","y","x");;
g:=GeneratorsOfAlgebraWithOne(A);;
z:=g[1];; y:=g[2];; x:=g[3];;
weights:=[1,1,1];;
KI_gp := [ z^4 +y*x*y*x - x*y*y*x - (3)*z*y*x*z,
x^3 + y*x*y - x*y*x, z*y*x-x*y*z + z*x*z ];;
KI_np:=GP2NPList(KI_gp);;
GB := SGrobnerTrunc(KI_np,10,weights,1);;
GBNP.ConfigPrint("z","y","x");
PrintNPList(GB);
Length(KI_gp);   Length(GB);
quit;
```

Command line call: `gap -b -q` < `lp1-10.gap` > `lp1-10.gap.res`.

MAGMA: `lp1-10.magma`

```
A<x,y,z>:=FreeAlgebra(Rationals(),3);
B := [ z^4 +y*x*y*x - x*y*y*x - 3*z*y*x*z, x^3 + y*x*y - x*y*x,
z*y*x-x*y*z + z*x*z ];
GB:=GroebnerBasis(B,10); // compute with the variation of F4
GB;
#B;  #GB;
quit;
```

In order to compute with Buchberger's algorithm instead of F4, one has to execute

```
GB:=GroebnerBasis(B,10:Faugere:=false);
```

Command line call: `magma` < `lp1-10.magma` > `lp1-10.magma.res`.

OPAL: `lp1-10.opal`

```
context A=PathAlgebra(Rationals,FreeGraph([x,y,z]));
assume A;
HBasis({z^4 +y*x*y*x - x*y*y*x - 3*z*y*x*z, x^3 + y*x*y - x*y*x,
z*y*x-x*y*z + z*x*z},10);
quit;
```

Command line call: `opal -o length lex` < `lp1-10.opal` > `lp1-10.opal.res`.

SINGULAR: `lp1-10.tst`

```
LIB "freegb.lib";
ring r = 0,(x,y,z),dp;
int d  = 10; // degree bound
def R  = makeLetterplaceRing(d); // create a letterplace ring
```

```
setring R;
ideal I = z(1)*z(2)*z(3)*z(4) + y(1)*x(2)*y(3)*x(4) - x(1)*y(2)*y(3)*x(4)
- 3*z(1)*y(2)*x(3)*z(4), x(1)*x(2)*x(3) + y(1)*x(2)*y(3) - x(1)*y(2)*x(3),
z(1)*y(2)*x(3)-x(1)*y(2)*z(3) + z(1)*x(2)*z(3);
option(redSB);option(redTail); // recommended options
ideal J = system("freegb",I,d,3); // main call
J; // print the generators
size(I); // the number of original generators
size(J); // the number of elements in Groebner basis
```

Command line call: `Singular -teq < lp1-10.tst > lp1-10.tst.res`

References

Apel, J., 2000. Computational ideal theory in finitely generated extension rings. Theoret. Comput. Sci. 244 (1–2), 1–33.

Apel, J., Lassner, W., 1988. An extension of Buchberger's algorithm and calculations in enveloping fields of Lie algebras. J. Symbolic Comput. 6 (2-3), 361–370.

Apel, J., 1988. Gröbnerbasen in nichtkommutativen Algebren und ihre Anwendung. Dissertation, Universität Leipzig.

Apel, J., Klaus, U., 1998. FELIX, a special computer algebra system for the computation in commutative and non-commutative rings and modules. Available from http://felix.hgb-leipzig.de/.

Aschenbrenner, M., Pong, W.Y., 2004. Orderings of monomial ideals. Fund. Math. 181, 27–74.

Backelin, J., et al.. 2006. The Gröbner basis calculator Bergman. Available from http://servus.math.su.se/bergman/.

Bergman, G., 1978. The diamond lemma for ring theory. Adv. Math. 29, 178–218.

Bokut, L., Klein, A., 1996. Serre relations and Gröbner–Shirshov bases for simple Lie algebras. I. Int. J. Algebra Comput. 6 (4), 389–400.

Bosma, W., Cannon, J., Playoust, C., 1997. The Magma algebra system. I: The user language. J. Symbolic Comput. 24 (3–4), 235–265.

Buchberger, B., 2006. An algorithm for finding the basis elements in the residue class ring modulo a zero dimensional polynomial ideal. In: Logic, Mathematics, and Computer Science: Interactions. J. Symbolic Comput. 41 (3–4), 475–511 (special issue).

Buchberger, B., 1970. An algorithmic criterion for the solvability of algebraic systems of equations. Aequationes Math. 4, 374–383 (in German).

Bueso, J., Gómez-Torrecillas, J., Verschoren, A., 2003. Algorithmic Methods in Non-commutative Algebra. Applications to Quantum Groups. Kluwer Academic Publishers.

Chyzak, F., Salvy, B., 1998. Non-commutative elimination in ore algebras proves multivariate identities. J. Symbolic Comput. 26 (2), 187–227.

Cohen, A.M., Gijsbers, D.A.H., 2003. GBNP, a Non-commutative Gröbner Bases Package for GAP 4. Available from http://www.win.tue.nl/~amc/pub/grobner/.

Cojocaru, S., Podoplelov, A., Ufnarovski, V., 1999. Non-commutative Gröbner bases and Anick's resolution. In: Dräxler, P. (Ed.), Computational Methods for Representations of Groups and Algebras (Proc. of the Euroconference in Essen, Germany, April 1997). Birkhäuser, pp. 29–60.

Doubilet, P., Rota, G.-C., Stein, J., 1974. On the foundations of combinatorial theory. IX. Combinatorial methods in invariant theory. Stud. Appl. Math 53, 185–216.

Drensky, V., La Scala, R., 2006. Gröbner bases of ideals invariant under endomorphisms. J. Symbolic Comput. 41, 835–846.

Eisenbud, D., 1995. Commutative Algebra with a view toward Algebraic Geometry. Springer-Verlag.

Evans, G., Wensley, C., 2007. Complete involutive rewriting systems. J. Symbolic Comput. 42 (11–12), 1034–1051. URL: http://dx.doi.org/10.1016/j.jsc.2007.07.005.

Feynman, R.P., 1951. An operator calculus having applications in quantum electrodynamics. Phys. Rev. 84, 108–128.

Green, D.J., 2003. Gröbner Bases and the Computation of Group Cohomology. In: Lecture Notes in Mathematics, vol. 1828. Springer.

Green, E., 1993. An introduction to noncommutative Gröbner bases. In: Fischer, K. (Ed.), Computational Algebra. Papers from the Mid-Atlantic Algebra Conference, May 1993. In: Dekker. Lect. Notes Pure Appl. Math., vol. 151. pp. 167–190.

Green, E., 2000. Multiplicative Bases, Gröbner Bases, and Right Gröbner Bases. J. Symbolic Comput. 29 (4–5).

Green, E., Heath, L., Keller, B., 1997. Opal: A System for Computing Noncommutative Gröbner Bases. In: RTA '97: Proceedings of the 8th International Conference on Rewriting Techniques and Applications. Springer, pp. 331–334.

Greuel, G.-M., Pfister, G., with contributions by Bachmann, O., Lossen, C., Schönemann, H., 2002. A SINGULAR Introduction to Commutative Algebra. Springer.

Greuel, G.-M., Pfister, G., Schönemann, H., 2006. Singular 3.0. A computer algebra system for polynomial computations. Centre for Computer Algebra, University of Kaiserslautern. Available from http://www.singular.uni-kl.de.

Helton, J.W., Stankus, M., 2001. NCGB 3.1, a Noncommutative Gröbner Basis Package for Mathematica. Available from http://www.math.ucsd.edu/~ncalg/.

Heyworth, A., 2000. Rewriting as a special case of non-commutative Gröbner basis theory. In: Atkinson, M.e.a. (Ed.), Computational and Geometric Aspects of Modern Algebra. Cambridge University Press, pp. 101–105.

Higman, G., 1952. Ordering by divisibility in abstract algebras. Proc. London Math. Soc. 2, 326–336.

Kandri-Rody, A., Weispfenning, V., 1990. Non-commutative Gröbner bases in algebras of solvable type. J. Symbolic Comput. 9 (1), 1–26.

Kang, S.-J., Lee, D.-I., Lee, K.-H., Park, H., 2007. Linear algebraic approach to Gröbner–Shirshov basis theory. J. Algebra 313 (2), 988–1004.

Keller, B.J., 1997. Algorithms and Orders for Finding Noncommutative Gröbner bases. Doctoral Thesis, Virginia Polytechnic Institute and State University.

Kredel, H., 1993. Solvable Polynomial Rings. Shaker.

Levandovskyy, V., 2005a. Non-commutative computer algebra for polynomial algebras: Gröbner bases, applications and implementation. Doctoral Thesis, Universität Kaiserslautern. Available from http://kluedo.ub.uni-kl.de/volltexte/2005/1883/.

Levandovskyy, V., 2005b. PBW Bases, Non-Degeneracy Conditions and Applications. In: Buchweitz, R.-O., Lenzing, H. (Eds.), Representation of Algebras and Related Topics (Proceedings of the ICRA X Conference), vol. 45. AMS. Fields Institute Communications, pp. 229–246.

Li, H., 2002. Noncommutative Gröbner Bases and Filtered-graded Transfer. Springer.

Mora, T., 1988. Seven variations on standard bases. Tech. Rep. 45, Univ. di Genova, Dip. di Mathematica, Preprint. URL: ftp://ftp.disi.unige.it/person/MoraF/PUBLICATIONS/7Varietions.tar.gz.

Mora, T., 1986. Gröbner bases for non-commutative polynomial rings. In: Proc. AAECC 3. In: Lect. N. Comp. Sci, vol. 229. pp. 353–362.

Mora, T., 1989. Groebner bases in non-commutative algebras. In: Proc. of the International Symposium on Symbolic and Algebraic Computation. ISSAC'88. In: LNCS, vol. 358. pp. 150–161.

Mora, T., 1994. An introduction to commutative and non-commutative Groebner bases. Theoret. Comput. Sci. 134, 131–173.

Nordbeck, P., 1998. On some basic applications of Gröbner bases in non-commutative polynomial rings. In: Buchberger, B., Winkler, F. (Eds.), Gröbner Bases and Applications. Cambridge University Press, pp. 463–472.

Pesch, M., 1998. Gröbner Bases in Skew Polynomial Rings. Shaker.

Pritchard, F.L., 1996. The ideal membership problem in non-commutative polynomial rings. J. Symbolic Comput. 22 (1), 27–48.

Reinert, B., 1995. On Gröbner bases in monoid and group rings. Doctoral Thesis, Universität Kaiserslautern.

Rosenmann, A., 1993. An algorithm for constructing Gröbner and free Schreier bases in free group algebras. J. Symbolic Comput. 16 (6), 523–549.

Shirshov, A.I., 1962. Some algorithmic problems for lie algebras. Siberian Mat. J. 3, 292–296.

Ufnarovski, V., 1995. Combinatorial and asymptotic methods of algebra. In: Kostrikin, A.I., Shafarevich, I.R. (Eds.), Algebra-VI. In: Encyclopedia of Mathematical Sciences, Springer.

Ufnarovski, V., 1998. Introduction to noncommutative Gröbner bases theory. In: Buchberger, B., Winkler, F. (Eds.), Gröbner Bases and Applications. Cambridge University Press, pp. 259–280.

Ufnarovski, V., 1989. On the use of graphs for calculating the basis, growth and Hilbert series of associative algebras. (Russian). Mat. Sb. 180, 1548–1560. Translation in Math. USSR-Sb., 68: 417–428, 1991.

Weispfenning, V., 1992. Finite Gröbner bases in non-Noetherian skew polynomial rings. In: Wang, P.S. (Ed.), Proc. of the International Symposium on Symbolic and Algebraic Computation. ISSAC'92. ACM Press, pp. 329–334.